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Analytical surface singularity distributions for flow about cylindrical bodies

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Abstract. In this paper exact analytical values of surface singularity distributions for two-dimensional potential flows are presented. A general formulation which provides an alternative to the Milne-Thomson theorem is presented for the circle, then distributions are obtained for flow about an ellipse. These solutions provide useful bench-mark test cases for examining the convergence properties in the development of some panel-method computer codes.

1. Introduction

The surface singularity distribution method or panel method, e.g. Hess and Smith [1], has long been a powerful and widely used approach for computing two-dimensional, incompressible potential flows about arbitrarily shaped bodies. The method can usefully be applied to flows about bluff bodies if separated flow regions are modelled. In early work, e.g. Gerrard [2], and Sarpkaya [3] regions of vorticity separated from the boundary layer of a circular cylinder were modelled by free potential line vortices. Then the Milne-Thomson circle theorem was used to obtain the corresponding image system inside the cylinder. In particular, Sarpkaya [3] used this approach to model the unsteady symmetric development of the wake behind an impulsively started circular cylinder. Benson et al. [4], adopting the same model (but with fixed initial location of the nascent vortices), computed the solution both by the image method and by a simple straight line constant source strength panel method with 128 panels. At each timestep, two nascent vortices were initially positioned about a panel length from the surface of the cylinder. After 30 timesteps the computed positions of the 60 external vortices calculated by both methods were graphically virtually indistinguishable.

Recently, computer algorithms have been improved by modelling the viscous diffusion process as well as the convection process, e.g. Chorin [5], Benson [6] and Benson et al. [7]. Then, as an alternative to the cloud-in-cell method, the convection of vorticity can still be computed using a simple potential line vortex method and for a general body profile this can be achieved using the panel method.

The present work was motivated by a desire to obtain analytical singularity distributions over the surface of two-dimensional bodies in incompressible flow to serve as bench-mark test cases for examining the improved rates of convergence of panel-method algorithms when, for example, higher-order effects such as body curvature are included. The panel method usually involves two main computational stages. Firstly, a linear system of equations is solved to obtain the discretised singularity distribution by applying the boundary condition at a number, N, of node points on the surface of the body. Then, secondly, the velocity field is calculated. While there are numerous analytical solutions for velocity distributions which allow testing of the final velocity output of panel-method algorithms, there appear to be few analytical solutions for the singularity distributions themselves which are calculated in the first stage of a panel-method algorithm. It is clearly advantageous to be able to analyse the accuracy of each of the main stages of a panel algorithm independently and so to direct improvements to the critical part of the computer code and also to identify any error cancellation occurring between the several parts of the code. This work provides analytical solutions for source/sink, vortex and dipole distributions for various flows about circles and ellipses.

By making use of Green's theorem, it has long been known, see Temple [8], that any irrotational field of flow in a bounded, simply connected region is the flow field of a distribution of singularities over the boundary (Green's 'equivalent stratum'). In such integral formulations, the integrand involves the unknown potential for the flow as a whole and so, as pointed out by Jeffreys and Jeffreys [9], 'direct application of these theorems to find the internal or external field is seldom possible', except in the cases of a sphere, circle or plane. The present work concerns two-dimensional flow only and so complex variables will be used for convenience. The approach adopted is similar to that used when proving the Milne-Thomson circle theorem [10]. In particular, it is assumed that $f(z)$ is the known complex potential for an unbounded flow in the absence of the body and that $f(z)$ is regular inside and on the region defined by the body surface. For example, a uniform stream and external vortices could contribute to $f(z)$. Then a singularity distribution on the body which ensures that the body surface is a streamline is sought in terms of $f(z)$. In general, the required distribution can be either a source/sink, a vortex or a dipole distribution or a combination of these.

The separated flow past a circular cylinder is still of widespread practical interest in itself. If the panel-method approach is adopted for the solution of the convective process in this flow, then the present work gives exact solutions for the singularity distribution and so it is unnecessary to solve for approximate values of the singularity distribution at discrete points. Thus, algorithms for the convective process could be made more accurate and economical by incorporating some of the results described below.

2. Mathematical formulation

A stationary two-dimensional body, B, is immersed in an incompressible inviscid fluid (see Fig. 1). Exploiting linearity, the velocity at a general point can be written in the form $\mathbf{Q} + \mathbf{u}$ where $\mathbf{Q}$ is the velocity in the absence of the body and $\mathbf{u}$ is the disturbance velocity (not necessarily small) due to the presence of the body. The potential flow problem is then to find a function φ, such that $\mathbf{u} = \text{grad}\ \varphi$, which satisfies the following conditions:

(i) $\nabla^2\varphi = 0$ exterior to the body,

(ii) $\text{grad}\ \varphi \rightarrow 0$ at infinity,

(iii) $\mathbf{n} \cdot \text{grad}\ \varphi = \partial\varphi/\partial n = -\mathbf{Q} \cdot \mathbf{n}$ on the surface S of the body, where $\mathbf{n}$ is the outwardly directed unit normal.

Let s be the arc length measured along the surface of the body and consider a source/sink distribution on the surface of local intensity $m(s)$ per unit length. Also let z and $z(s)$ be complex numbers representing a general point P, exterior to B, and a general point on the

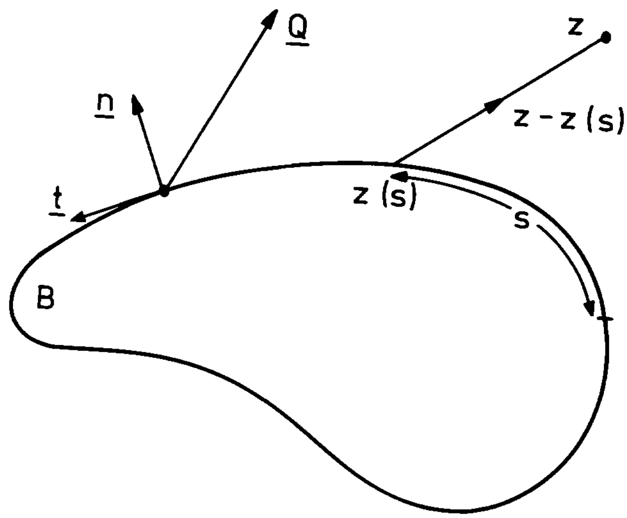

Fig. 1. Body B, placed in an otherwise unbounded potential flow.

surface, respectively. Then the potential

$$\varphi = \frac{1}{2\pi} \oint_B m(s) \log|z - z(s)| \, ds \tag{2.1}$$

satisfies conditions (i) and (ii) exactly. Let $z \to z(s_1)$ so that P lies on the surface at $s = s_1$. Then condition (iii) requires that

$$\frac{1}{2\pi} \oint_B m(s) \frac{\partial}{\partial n} [\log|z(s_1) - z(s)|] \, ds = -(\mathbf{Q} \cdot \mathbf{n})_{s_1} . \tag{2.2}$$

When $s \to s_1$ the integrand in (2.2) becomes singular and so the left-hand side of (2.2) can be rewritten as the sum of an integral of Cauchy principal value type and the term $m(s_1)/2$, see Jaswon and Symm [11],

$$\frac{m(s_1)}{2} + \frac{1}{2\pi} ⨍_B m(s) \frac{\partial}{\partial n} [\log|z(s_1) - z(s)|] \, ds = -(\mathbf{Q} \cdot \mathbf{n})_{s_1} \tag{2.3}$$

This is a linear Fredholm integral equation of the second kind. Here, $(2\pi)^{-1}\partial[\log|z(s_1) - z(s)|]/\partial n$ is the component of velocity normal to the body surface induced at $s = s_1$ by a unit source at s, which in the case of a circular cylinder of radius a, equals $(4\pi a)^{-1}$. Then the integrand in (2.3) is m times a constant. For a closed body the integral of m is zero. Hence the Cauchy principal value in (2.3) is zero and so

$$m(s_1) = -2(\mathbf{Q} \cdot \mathbf{n})_{s_1} . \tag{2.4}$$

Now let $f(z)$ be the complex potential of the flow in the absence of the circle. If the circle of radius a is centred at the origin and $s = a\theta$, where θ is the azimuthal angle measured in the anti-clockwise direction from the positive real axis, then

$$(\mathbf{Q} \cdot \mathbf{n})_s = \mathrm{Re}[e^{i\theta} f'(a\, e^{i\theta})] \tag{2.5}$$

where a dash denotes differentiation with respect to z. Hence, from (2.4) and (2.5), source/sink distributions for various external flows can be obtained as presented in Table 1.

Table 1. Singularity distributions for various flows past a circular cylinder

External flow	Source/sink distribution	Vortex distribution
Uniform stream, U along the x-axis	$-2U\cos\theta$	$-2U\sin\theta$
Point source of strength M at $(d,0)$	$\dfrac{M(d\cos\theta-a)}{\pi(a^2+d^2-2ad\cos\theta)}$	$\dfrac{Md\sin\theta}{\pi(a^2+d^2-2ad\cos\theta)}$
Point vortex of strength Γ at $(d,0)$. (Clockwise)	$\dfrac{-\Gamma d\sin\theta}{\pi(a^2+d^2-2ad\cos\theta)}$	$\dfrac{\Gamma(d\cos\theta-a)}{\pi(a^2+d^2-2ad\cos\theta)}$
Doublet of strength M at $(d,0)$ aligned along the x-axis	$\dfrac{M\{(a^2+d^2)\cos\theta-2ad\}}{\pi(a^2+d^2-2ad\cos\theta)^2}$	$\dfrac{M(d^2-a^2)\sin\theta}{\pi(a^2+d^2-2ad\cos\theta)^2}$
Circulation, K about the cylinder	—	$\dfrac{K}{2\pi a}$

N.B. If the location of the external singularity is $d\exp(i\alpha)$, then in the above expressions, replace θ by $\theta-\alpha$.

Also from (2.4) and (2.5),

$$m(\theta)=-\left\{\frac{1}{\zeta}\bar{f}'\left(\frac{a}{\zeta}\right)+\zeta f'(a\zeta)\right\}$$

where $\zeta=\exp(i\theta)$, bars denote complex conjugates and $\bar{f}(z)=\overline{f(\bar{z})}$. Hence, when the cylinder is present, the complex potential is

$$w(z)=f(z)-\frac{a}{2\pi i}\oint_C\left\{\frac{1}{\zeta^2}\bar{f}'\left(\frac{a}{\zeta}\right)+f'(a\zeta)\right\}\log(z-a\zeta)\,d\zeta \tag{2.6}$$

where C is the unit-radius circle, $|\zeta|=1$. Differentiating (2.6) gives

$$\begin{aligned}\frac{dw}{dz}&=f'(z)+\frac{1}{2\pi i}\oint_C\frac{\dfrac{1}{\zeta^2}\bar{f}'\left(\dfrac{a}{\zeta}\right)+f'(a\zeta)}{\zeta-\dfrac{z}{a}}\,d\zeta \\ &=f'(z)+(\text{Sum of residues of poles inside } C)\,.\end{aligned} \tag{2.7}$$

Now let S_1 be the sum of the residues of poles of

$$\left\{\frac{1}{\zeta^2}\bar{f}'\left(\frac{a}{\zeta}\right)+f'(a\zeta)\right\}\Big/\left(\zeta-\frac{z}{a}\right) \tag{2.8}$$

which lie inside C when $|z|>a$. Then, in Appendix 1, it is shown that

$$S_1=-(a/z)^2\bar{f}'(a^2/z)\,, \tag{2.9}$$

so giving dw/dz in accordance with Milne-Thomson's circle theorem. If, however, $|z|<a$, the velocity inside the circular cylinder $|z|=a$ given by (2.7) will be different from that arising from the circle theorem. When $|z|<a$, the function (2.8) has an additional pole of order one inside C at $\zeta=z/a$. The residue of this pole is given by

$$S_2 = (a/z)^2\bar{f}'(a^2/z) + f'(z)\,. \tag{2.10}$$

Hence, when $|z| < a$,

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f'(z) + S_1 + S_2 = 2f'(z)\,. \tag{2.11}$$

A simple example for a particular choice of $f(z)$ is given in Appendix 2.

It is clear that circulation about the cylinder cannot be obtained by means of a source/sink singularity distribution. An alternative approach, however, is to account for the presence of the cylinder by a vortex distribution, $\Gamma(\theta)$ say, on the surface. Here $\Gamma(\theta)$ is positive for circulation in the anti-clockwise sense. Then, corresponding to equations (2.4) and (2.5) we have

$$\Gamma(\theta) = 2(\mathbf{Q}\cdot\mathbf{t})_\theta = -2\,\mathrm{Im}[\mathrm{e}^{\mathrm{i}\theta} f'(a\,\mathrm{e}^{\mathrm{i}\theta})]\,, \tag{2.12}$$

where $\mathbf{t}$ is a unit vector tangential to the circle in the direction of θ increasing. Hence vortex distributions for various external flows can be obtained as presented in Table 1. Note that the distribution corresponding to a circulation about the cylinder is included. If ζ and C are as defined earlier, then

$$\Gamma(\theta) = \mathrm{i}\left\{\zeta f'(a\zeta) - \frac{1}{\zeta}\,\bar{f}'\left(\frac{a}{\zeta}\right)\right\} \tag{2.13}$$

and

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f'(z) + \frac{1}{2\pi\mathrm{i}}\oint_C \frac{\dfrac{1}{\zeta^2}\,\bar{f}'\left(\dfrac{a}{\zeta}\right) - f'(a\zeta)}{\zeta - \dfrac{z}{a}}\,\mathrm{d}\zeta\,. \tag{2.14}$$

Now $f'(a\zeta)$ is regular inside C and so is $1/(\zeta - z/a)$ when $|z| > a$. Hence, when $|z| > a$, the integral in (2.14) has the same value as the integral in (2.7), namely S_1 as given by equation (2.9). When $|z| < a$, however, the residue, S_2 of the additional pole inside C at $\zeta = z/a$ is given by

$$S_2 = (a/z)^2\bar{f}'(a^2/z) - f'(z)\,.$$

Hence, when $|z| < a$, $\mathrm{d}w/\mathrm{d}z = 0$ and the fluid is at rest inside the cylinder.

Equivalent dipole distributions can be obtained by integrating equation (2.6) by parts. Then

$$\begin{aligned} w(z) &= f(z) + \frac{a}{2\pi\mathrm{i}}\oint_C \frac{\bar{f}(a/\zeta) - f(a\zeta)}{z - a\zeta}\,\mathrm{d}\zeta \\ &= f(z) + \frac{1}{2\pi}\int_0^{2\pi} \frac{-2\,\mathrm{Im}[f(a\,\mathrm{e}^{\mathrm{i}\theta})]}{z - a\,\mathrm{e}^{\mathrm{i}\theta}}\,\mathrm{e}^{\mathrm{i}(\theta+\frac{\pi}{2})}\,a\,\mathrm{d}\theta\,. \end{aligned}$$

The integral now represents a doublet distribution of strength $M(\theta)$ per unit length where

$$M(\theta) = 2\,\mathrm{Im}[f(a\,\mathrm{e}^{\mathrm{i}\theta})]\,,$$

and the axis of the doublets is tangential to the cylinder. In general, a source-strength distribution m is simply related to the equivalent doublet distribution M by the equation

$$m = -\frac{\mathrm{d}M}{\mathrm{d}s}\,, \tag{2.15}$$

which for a circular cylinder reduces to

$$m(\theta) = -\frac{1}{a}\frac{\mathrm{d}M}{\mathrm{d}\theta}\,.$$

Similarly, starting with a vortex distribution, an equivalent doublet distribution, $G(\theta)$ can be obtained where

$$G(\theta) = -2\,\mathrm{Re}[f(a\,\mathrm{e}^{\mathrm{i}\theta})]\,,$$

and the axis of the doublets is normal to the surface of the cylinder.

To avoid the singularity in (2.2), there may be computational advantages in locating singularity distributions inside the body, B instead of on the surface of B, especially in the case of bluff bodies. As a simple analytical example, if B is a circle of radius a in a uniform stream U, an equivalent source/sink distribution on the concentric circle of radius r ($< a$) is given by

$$m(\theta) = -2U(a/r)^2\cos\theta\,.$$

For an elliptical cylinder with Cartesian coordinates ($a\cos\varphi$, $b\sin\varphi$) say, there is no easy way to get m from equation (2.3). Although the integral of m about the ellipse is zero, the integrand in (2.3) is no longer m times a constant and so the Cauchy principal value is not zero. Moreover, conformal transformation of the singularity distribution on a circle to an ellipse will yield a correct result but not in the form of a simple singularity distribution over the surface of the ellipse. However, for a uniform stream along the x-axis, the case of the circular cylinder suggests seeking a source/sink distribution of the form

$$m(\varphi) = U\lambda\cos\varphi/(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}\,.$$

Then it is found that (2.3) is satisfied if

$$\lambda = -b(a+b)/a.$$

Then the flow outside the ellipse is the same as obtained by conformal transformation, whereas the flow inside the ellipse is a uniform stream, $U(a+b)/a$ along the x-axis. This follows from the equation corresponding to (2.7):

$$\frac{\mathrm{d}w}{\mathrm{d}z} = U + \frac{Ub}{2\pi\mathrm{i}a}\oint_C \frac{(\zeta^2+1)\,\mathrm{d}\zeta}{\zeta(\zeta-z_1)(\zeta-z_2)}\,, \tag{2.16}$$

where C is the unit radius circle, $|\zeta| = 1$ and where z_1 and z_2 are the roots of

$$\zeta^2 - \left(\frac{2z}{a+b}\right)\zeta + \frac{a-b}{a+b} = 0\,.$$

If z lies outside the ellipse then the above contour integral has two poles inside C, at $\zeta = 0$ and $\zeta = z_2$ whereas if z lies inside the ellipse, there is an additional pole inside C at $\zeta = z_1$.

The corresponding distribution of dipoles with axes tangential to the ellipse is obtained from equation (2.15), (or alternatively from (2.16)) as follows:

$$M(\varphi) = -U\lambda \sin\varphi .$$

It may be noted that these results apply to an elliptical cylinder whose major axis is normal to the free-stream direction if $b > a$.

If a uniform stream is at a general incidence α to the ellipse, the following vortex distribution is obtained,

$$\Gamma(\varphi) = -U(a+b)\sin(\varphi - \alpha)/(a^2 \sin^2\varphi + b^2\cos^2\varphi)^{1/2} . \tag{2.17}$$

If $b \to 0$, the vortex distributions over the upper and lower surfaces of the ellipse coalesce. Then, in the limit, (2.17) reduces to

$$\Gamma = 2Ux\sin\alpha/(a^2 - x^2)^{1/2} ,$$

the usual result for a flat plate at incidence.

The vortex distribution corresponding to a circulation K about the ellipse is given by

$$\Gamma(\varphi) = K/\{2\pi(a^2\sin^2\varphi + b^2\cos^2\varphi)^{1/2}\} . \tag{2.18}$$

Distributions (2.17) and (2.18) both give zero velocity inside the ellipse. Corresponding doublet distributions can be obtained.

3. Conclusions

The effect on a general unbounded two-dimensional potential incompressible flow, with undisturbed vector velocity $\mathbf{Q}$, with no singularities within the region $|z| \leq a$, has been considered when a circular cylinder, $|z| = a$, is introduced into the flowfield. Instead of applying the Milne-Thomson circle theorem to obtain an image system inside the circle, it has been shown that an equivalent analytical singularity distribution on the surface of the cylinder can be obtained. This can be either a source/sink distribution (if there is no circulation about the cylinder), given by equation (2.4) or a vortex-sheet distribution given by (2.12). Hence

$$\mathbf{Q}_s = \tfrac{1}{2}[\Gamma(\theta)\mathbf{t} - m(\theta)\mathbf{n}] \tag{3.1}$$

where $\mathbf{t}$ and $\mathbf{n}$ are unit vectors tangential and normal to the cylinder as defined earlier. If $f(z)$ is the complex potential of the flow before the introduction of the cylinder, then (3.1) can be expressed in the following complex form,

$$m(\theta) + \mathrm{i}\Gamma(\theta) = -2\,\mathrm{e}^{\mathrm{i}\theta} f'(a\,\mathrm{e}^{\mathrm{i}\theta}) . \tag{3.2}$$

The singularity distribution on the cylinder can also be a combination of a source/sink distribution and a vortex distribution of the general form

$$km(\theta) + (1-k)\mathrm{i}\Gamma(\theta)$$

where k is a real constant. Then the corresponding general form of the complex potential is given by

$$\frac{\mathrm{d}w}{\mathrm{d}z} = f'(z) + \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\frac{1}{\zeta^2} \bar{f}'\left(\frac{a}{\zeta}\right) + (2k-1) f'(a\zeta)}{\zeta - \frac{z}{a}} \, \mathrm{d}\zeta \tag{3.3}$$

where C is the unit circle $|\zeta| = 1$. For a distribution entirely of sources and sinks, $k = 1$ whereas for a distribution entirely of vortices, $k = 0$. Equivalently, these monopole distributions can be expressed as dipole distributions.

For models of separated flows past circular cylinders of the type studied by Sarpkaya [3] and Benson et al. [4], the present results give the corresponding singularity distribution on the surface if a panel approach is adopted. For example, if there are n vortices in the wake of the cylinder and the typical ith vortex has strength Γ_i and is located at $d_i \exp(\mathrm{i}\alpha_i)$ then the corresponding source/sink distribution is

$$m(\theta) = -2U \cos\theta - \sum_{i=1}^{n} \frac{\Gamma_i d_i \sin(\theta - \alpha_i)}{\pi(a^2 + d_i^2 - 2ad_i \cos(\theta - \alpha_i))} \, . \tag{3.4}$$

If a vortex is close to the circle then the distribution on the circle behaves like $1/\varepsilon$ where ε is the distance between the vortex and the surface. Then a large number of panels may be required to compute a solution comparable to that of the Milne-Thomson approach. The numerical example of Benson et al. [4] suggests that close agreement between the two methods is obtained if vortices are at or beyond one panel length from the body surface. The error introduced near the surface by the discretisation of (3.4) depends on the order of the panel method adopted as discussed below.

Bellamy-Knights et al. [12] compare the analytical singularity distributions presented here with the distributions predicted by panel methods of different order and for $N = 16$, 32, 64 and 128 where N is the number of panels. It is found numerically that for the 'zeroth'-order panel method (i.e. straight-line panels of constant strength), the singularity strength and external velocity field converge only as $1/N$. If, however, panel curvature is taken into account (maintaining constant panel strength) then the singularity strength and external velocity field converge as $1/N^2$. These results led Bellamy-Knights et al. [12] to re-examine the work of Hess [13]. Analysis of the velocity induced at the control point of a panel by that panel itself verified that panel curvature should be taken into account to allow the computed strength distribution and external velocity field to converge as $1/N^2$. To compute the velocity distribution on the surface of the body to order $1/N^2$, it is additionally required to take panel strength variation into account. This analysis applies to bodies of arbitrary shape and it suggests that the panel method of Shaw [14], which takes account of panel curvature and strength variation, everywhere computes velocities which converge as $1/N^2$.

Appendix 1

The following integral, I, will now be evaluated for $|z| > a$:

$$I = \frac{1}{2\pi \mathrm{i}} \oint_C \left[\frac{1}{\zeta^2} \bar{f}'\left(\frac{a}{\zeta}\right) + f'(a\zeta)\right] \Big/ \left[\zeta - \frac{z}{a}\right] \mathrm{d}\zeta$$

where C is the unit circle $|\zeta| = 1$ and $f(z)$ is regular inside and on the circle $|z| = a$.

First note that $f'(a\zeta)$ and $(\zeta - z/a)$ are both regular inside C and so I reduces to

$$I = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\bar{f}'(a/\zeta)\,\mathrm{d}\zeta}{\zeta^2\left(\zeta - \frac{z}{a}\right)}\,.$$

On applying the substitution $\eta = 1/\zeta$, I is transformed into a contour integral in the η-plane, i.e.,

$$I = \frac{-1}{2\pi \mathrm{i}} \oint_D \frac{a\eta \bar{f}'(a\eta)\,\mathrm{d}\eta}{z\left(\eta - \frac{a}{z}\right)}$$

where D is the unit circle $|\eta| = 1$. Now $f'(a\eta)$ is regular inside the circle $|\eta| = 1$. Since $|z| > a$, there is one simple pole inside $|\eta| = 1$ at $\eta = a/z$. The residue of the pole at $\eta = a/z$ is $(a/z)^2\bar{f}'(a^2/z)$. Hence

$$I = -(a/z)^2\bar{f}'(a^2/z)\,.$$

Appendix 2

As an illustrative example, consider a point vortex of strength Γ (in the clockwise sense) at the point A $(d, 0)$ where $d > a$. Then, with reference to Fig. 2, the magnitude, $|\mathbf{Q}|$ of the velocity induced at the general point B of the circle of radius a is given by $\Gamma/(2\pi \mathrm{AB})$. Hence, using equation (2.4), the corresponding source distribution is

$$m(\theta) = -2\mathbf{Q}\cdot\mathbf{n} = -\frac{\Gamma \sin\beta}{\pi \mathrm{AB}} = -\frac{\Gamma d \sin\theta}{\pi(a^2 + d^2 - 2ad\cos\theta)}\,.$$

The complex potential, W for this source distribution on the circle $|z| = a$ is given by

$$W = \int_0^{2\pi} \frac{m(\theta)}{2\pi} \log(z - a\,\mathrm{e}^{\mathrm{i}\theta})a\,\mathrm{d}\theta\,.$$

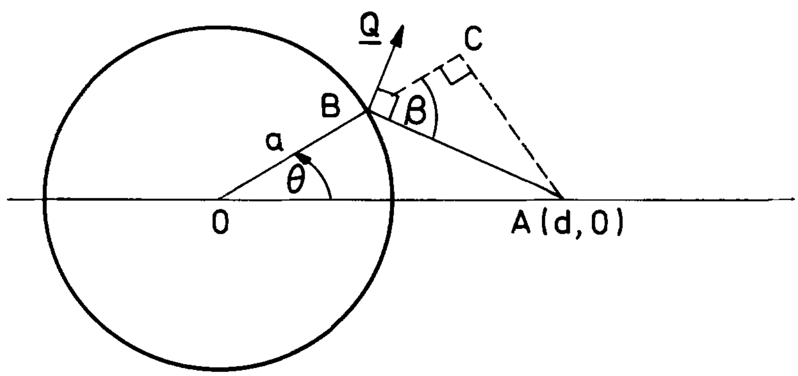

Fig. 2. Geometrical definitions for a vortex outside a circle.

Then, putting $\zeta = \exp(\mathrm{i}\theta)$, it follows that

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\Gamma}{4\pi^2 a} \oint_C \frac{(\zeta^2 - 1)\,\mathrm{d}\zeta}{\zeta\left(\zeta - \frac{a}{d}\right)\left(\zeta - \frac{z}{a}\right)\left(\zeta - \frac{d}{a}\right)}$$

where C is the unit radius circle $|\zeta| = 1$.

The integrand has four simple poles, at $\zeta = 0$, $\zeta = a/d$, $\zeta = z/a$ and $\zeta = d/a$. The first two poles lie inside C, the third pole lies either inside or outside C and the fourth pole lies outside C. Let the residues of the first three poles be R_1, R_2 and R_3 respectively. Then

$$R_1 = \frac{a}{z}, \qquad R_2 = -\frac{a}{z - \dfrac{a^2}{d}}, \qquad R_3 = \frac{a}{z-d} + \frac{a}{z - \dfrac{a^2}{d}} - \frac{a}{z}.$$

If $|z| > a$, then only the first two poles lie inside C. Hence

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\Gamma}{4\pi^2 a} \cdot 2\pi\mathrm{i} \cdot (R_1 + R_2) = \frac{\mathrm{i}\Gamma}{2\pi}\left\{\frac{1}{z} - \frac{1}{z - \dfrac{a^2}{d}}\right\}.$$

This corresponds to a clockwise rotating vortex at $z = 0$ and an anti-clockwise vortex at $z = a^2/d$. Hence when $|z| > a$, the source distribution gives the same flow as obtained from the image vortex system given by the Milne-Thomson circle theorem.

If $|z| < a$, then the first three poles lie inside C. Hence

$$\frac{\mathrm{d}W}{\mathrm{d}z} = \frac{\Gamma}{4\pi^2 a} \cdot 2\pi\mathrm{i} \cdot (R_1 + R_2 + R_3) = \frac{\mathrm{i}\Gamma}{2\pi}\,\frac{1}{z-d}.$$

The complex potential w due to the vortex at A and the source/sink distribution is given by

$$w = \frac{\mathrm{i}\Gamma}{2\pi}\log(z-d) + W \Rightarrow \frac{\mathrm{d}w}{\mathrm{d}z} = 2\,\frac{\mathrm{i}\Gamma}{2\pi}\,\frac{1}{z-d}.$$

Thus equation (2.11) is confirmed.

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